



An Algorithm for Solving Variable Coefficient Hyperbolic Problems in a Semi-Infinite Medium

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Abstract—This paper provides an algorithm for constructing numerical solutions of variable coefficient problems in a semi-infinite medium. Using sine Fourier transform an integral expression of the solution is found. Then numerical integration and the numerical solution of certain underlying differential equations are used to establish the algorithm. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Wave equations with space variable coefficient are frequent in optics [1] and microwave heating processes [2]. Mixed problems for this equation has been studied in [3] using separation of variables method. In this paper, we consider the problem

$$u_{xx}(x, t) = c^2(x)u_{tt}(x, t) + f(t), \quad x > 0, \quad t > 0, \quad (1)$$

$$u(0, t) = p(t), \quad t \geq 0, \quad (2)$$

$$u_x(0, t) = q(t), \quad t \geq 0, \quad (3)$$

$$u(x, 0) = 0, \quad x \geq 0, \quad (4)$$

where

$$c(x) \text{ is continuously differentiable and positive,} \quad (5)$$

$$f \text{ is three times differentiable and } f^{(i)} \text{ is Lebesgue integrable for } 0 \leq i \leq 3, \quad (6)$$

$$p \text{ is four times differentiable and } p^{(i)} \text{ is Lebesgue integrable for } 0 \leq i \leq 4, \quad (7)$$

and

$$q \text{ is three times differentiable and } q^{(i)} \text{ is Lebesgue integrable for } 0 \leq i \leq 3. \quad (8)$$

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The aim of this paper is to obtain a constructive algorithm for solving problem (1)–(4) under hypotheses (5)–(8). In Section 2, an integral solution of the problem is obtained using sine Fourier transforms. Section 3 deals with the construction of numerical solutions by truncating the infinite integral expression of the exact solution as well as numerical integration and the numerical solution of certain underlying differential equation using Stormer method. An algorithm which permits the symbolic computation of the numerical solution at the point (X, T) is included.

Throughout this paper, we denote by $L^1 = L^1([0, \infty[)$ the set of all Lebesgue integrable real-valued functions defined on the interval $[0, \infty[$. If h lies in L^1 , denote by $\mathcal{F}_s[h](w)$ the sine Fourier transform of h ,

$$H(w) = \mathcal{F}_s[h](w) = \int_0^\infty h(x) \sin(wx) dx, \quad w \in [0, \infty[. \quad (9)$$

We recall some properties of the sine Fourier transform, see [4]. If h is differentiable and $h^{(i)}$ lies in L^1 , where i is a positive integer, then

$$H(w) = \mathcal{F}_s[h](w) = O(w^{-i}), \quad \text{as } w \rightarrow \infty. \quad (10)$$

If h is twice differentiable and $h^{(i)}$ lies in L^1 for $i = 0, 1, 2$, then

$$\mathcal{F}_s[h''](w) = -w^2 \mathcal{F}_s[h](w) + w h(0). \quad (11)$$

Note that by hypotheses (6)–(8) and (10), $\mathcal{F}_s[p](w) = P(w)$, $\mathcal{F}_s[q](w) = Q(w)$, and $\mathcal{F}_s[f](w) = F(w)$ satisfy

$$|P(w)| = O(w^{-4}), \quad |Q(w)| = O(w^{-3}), \quad |F(w)| = O(w^{-3}), \quad \text{as } w \rightarrow \infty. \quad (12)$$

2. EXACT INTEGRAL SOLUTION

Let us assume for the moment that problem (1)–(4) admits a solution $u(x, t)$ such that

$$u(x, \cdot), u_x(x, \cdot), u_t(x, \cdot), u_{xx}(x, \cdot), u_{tt}(x, \cdot) \text{ and are in } L^1([0, \infty[), \quad (13)$$

and let

$$\begin{aligned} U(x)(w) &= \mathcal{F}_s\{u(x, \cdot)\}(w), & F(w) &= \mathcal{F}_s\{f(\cdot)\}(w), \\ P(w) &= \mathcal{F}_s\{p(\cdot)\}(w), & Q(w) &= \mathcal{F}_s\{q(\cdot)\}(w). \end{aligned} \quad (14)$$

By property (10) and condition $u(x, 0) = 0$, it follows that

$$\mathcal{F}_s\{u_{xx}(x, \cdot)\}(w) = -w^2 \mathcal{F}_s\{u(x, \cdot)\}(w) = -w^2 U(x)(w). \quad (15)$$

By applying sine Fourier transform to problem (1)–(4) and taking into account (14), (15), one gets that $U(x)(w)$ satisfies the parametric differential equation

$$\begin{aligned} \frac{d^2}{dx^2} U(x)(w) &= -c^2(x) w^2 U(x)(w) + F(w), & x &\geq 0, \\ U(0)(w) &= P(w), & \frac{d}{dx} U(0)(w) &= Q(w). \end{aligned} \quad (16)$$

Let $\{\varphi_1(x, w), \varphi_2(x, w)\}$ be the fundamental set of solutions of equation

$$y'' = -c^2(x) w^2 y, \quad x \geq 0, \quad (17)$$

satisfying

$$\varphi_1(0, w) = 1, \quad \frac{d}{dx} \varphi_1(0, w) = 0, \quad \varphi_2(0, w) = 0, \quad \frac{d}{dx} \varphi_2(0, w) = 1. \quad (18)$$

Note that by (18), the Wronskian

$$W(x, w) = \det \begin{bmatrix} \varphi_1(x, w) & \varphi_2(x, w) \\ \frac{d}{dx} \varphi_1(x, w) & \frac{d}{dx} \varphi_2(x, w) \end{bmatrix} \quad (19)$$

satisfies $\frac{d}{dx} W(x, w) = 0$ for all $x \geq 0, w \geq 0$. Hence,

$$W(x, w) = 1, \quad x \geq 0, \quad w \geq 0. \quad (20)$$

Since $c(x)$ and $F(x)(w)$ are continuous functions of the variable x for a fixed value of w , the variation of the parameters method yields the unique solution of (16) given by

$$U(x)(w) = \varphi_1(x, w)M_1(x, w) + \varphi_2(x, w)M_2(x, w), \quad x \geq 0, \quad w \geq 0, \quad (21)$$

$$M_1(x, w) = P(w) - F(w) \int_0^x \varphi_2(s, w) ds, \quad M_2(x, w) = Q(w) + F(w) \int_0^x \varphi_1(s, w) ds. \quad (22)$$

By the sine Fourier inversion formula, for a fixed point (X, T) , with $X > 0, T > 0$, one gets the candidate solution

$$u(X, T) = \mathcal{F}_s^{-1}\{U(X)(w)\} = \frac{2}{\pi} \int_0^\infty \{\varphi_1(X, w)M_1(X, w) + \varphi_2(X, w)M_2(X, w)\} \sin(wT) dw. \quad (23)$$

Now we prove that under hypotheses (5)–(8), $u(X, T)$ defined by (23) is a rigorous solution of problem (1)–(4).

First, by Theorem 4.1 of [5, p. 9] it follows that

$$|\varphi_1(x, w)| \leq L_0, \quad 0 \leq x \leq X, \quad \left| \frac{d}{dx} \varphi_1(x, w) \right| \leq |w|L_0, \quad 0 \leq x \leq X, \quad (24)$$

$$|\varphi_2(x, w)| \leq \frac{L_0}{|w|c(0)}, \quad w \neq 0, \quad 0 \leq x \leq X, \quad \left| \frac{d}{dx} \varphi_2(x, w) \right| \leq \frac{L_0}{c(0)}, \quad 0 \leq x \leq X, \quad (25)$$

$$C_1(X) = \max \{2c(x)c'(x), 0 \leq x \leq X\}, \quad (26)$$

$$C_2(X) = \min \{c^2(x), 0 \leq x \leq X\}, \quad (27)$$

$$L_0(X) = \exp \left(\frac{XC_1(X)}{2C_2(X)} \right). \quad (28)$$

By hypotheses (6)–(8) together with (21), (22), (24), (25), and (11), it follows that

$$\varphi_1(x, w)M_1(x, w) + \varphi_2(x, w)M_2(x, w) = O(|w|^{-2}), \quad \text{as } w \rightarrow \infty \quad (29)$$

uniformly for $0 \leq x \leq X$.

Hence, $u(X, T)$ given by (23) is well defined. Otherwise, by (24), (25), and (28), it follows that

$$M_1(x, w) = O(|w|^{-4}), \quad M_2(x, w) = O(|w|^{-3}), \quad \text{as } w \rightarrow \infty \text{ uniformly for } 0 \leq x \leq X. \quad (30)$$

By (24), (25), and (30), one gets

$$\begin{aligned} & \frac{\partial}{\partial x} \{\varphi_1(x, w)M_1(x, w) + \varphi_2(x, w)M_2(x, w)\} \\ &= \frac{\partial}{\partial x} \varphi_1(x, w)M_1(x, w) + \frac{\partial}{\partial x} \varphi_2(x, w)M_2(x, w) \\ &= O(w^{-3}), \quad \text{as } w \rightarrow \infty \text{ uniformly for } 0 \leq x \leq X. \end{aligned} \quad (31)$$

By (20), (29), and (30) it follows that

$$\begin{aligned}
 & \frac{\partial^2}{\partial x^2} \{ \varphi_1(x, w) M_1(x, w) + \varphi_2(x, w) M_2(x, w) \} \\
 &= \frac{\partial^2}{\partial x^2} \varphi_1(x, w) M_1(x, w) - \frac{\partial}{\partial x} \varphi_1(x, w) \varphi_2(x, w) F(w) \\
 & \quad + \frac{\partial^2}{\partial x^2} \varphi_2(x, w) M_2(x, w) + \frac{\partial}{\partial x} \varphi_2(x, w) \varphi_1(x, w) F(w) \\
 &= c^2(x) w^2 \{ \varphi_1(x, w) M_1(x, w) + \varphi_2(x, w) M_2(x, w) \} + F(w) \\
 &= O(w^{-2}), \quad \text{as } w \rightarrow \infty \text{ uniformly for } 0 \leq x \leq X.
 \end{aligned} \tag{32}$$

By (29), (30), we have

$$\begin{aligned}
 & \frac{\partial^2}{\partial t^2} \{ \sin(wt) (\varphi_1(x, w) M_1(x, w) + \varphi_2(x, w) M_2(x, w)) \} \\
 &= w^2 \sin(wt) \{ \varphi_1(x, w) M_1(x, w) + \varphi_2(x, w) M_2(x, w) \} \\
 &= O(w^{-2}), \quad \text{as } w \rightarrow \infty \text{ uniformly for } 0 \leq x \leq X, \ 0 \leq t \leq T.
 \end{aligned} \tag{33}$$

By (29), (31)–(33) and the derivation theorem of improper integrals, see Theorem 8.11.2 in [6, p. 174], one gets that $u(X, T)$ given by (23) is well defined and it is twice partially differentiable with respect to both variables x, t , and by the sine Fourier inversion theorem [4], it follows that

$$\begin{aligned}
 & u_{xx}(X, T) - c^2(X) u_{tt}(X, T) \\
 &= \frac{2}{\pi} \int_0^\infty \{ \varphi_1(X, w) M_1(X, w) + \varphi_2(X, w) M_2(X, w) \} \{ c^2(X) w^2 - c^2(X) w^2 \} \sin(wT) dw \\
 & \quad + \frac{2}{\pi} \int_0^\infty F(w) \sin(wT) dw = f(T), \quad X > 0, \ T > 0.
 \end{aligned}$$

By (29), it follows that $u(X, T)$ is continuous and taking limits in (23) as $x \rightarrow 0$, and using the inversion theorem for the sine Fourier transform it follows that

$$\lim_{x \rightarrow 0} u(X, T) = \frac{2}{\pi} \int_0^\infty \varphi_1(0, w) M_1(0, w) \sin(wT) dw = \frac{2}{\pi} \int_0^\infty P(w) \sin(wT) dw = P(T).$$

By (31), one gets that u is continuously partially differentiable with respect to the variable x . Taking limits in the expression

$$u_x(X, T) = \frac{2}{\pi} \int_0^\infty \left\{ \frac{\partial}{\partial x} \varphi_1(X, w) M_1(X, w) + \frac{\partial}{\partial x} \varphi_2(X, w) M_2(X, w) \right\} \sin(wT) dw$$

and using the inversion theorem for the sine Fourier transform it follows that

$$\begin{aligned}
 \lim_{x \rightarrow 0} u_x(X, T) &= \frac{2}{\pi} \int_0^\infty \left\{ \frac{\partial}{\partial x} \varphi_1(0, w) M_1(0, w) + \frac{\partial}{\partial x} \varphi_2(0, w) M_2(0, w) \right\} \sin(wT) dw \\
 &= \frac{2}{\pi} \int_0^\infty Q(w) \sin(wT) dw = q(T).
 \end{aligned}$$

Summarizing the following result has been established.

THEOREM 2.1. Assume that c, p, q , and f satisfy hypotheses (5)–(8), respectively. Then $u(X, T)$ defined by (23) is an exact solution of problem (1)–(4) for $X > 0, T > 0$.

3. NUMERICAL SOLUTION

Expression (23) has two drawbacks from the computational point of view. First, the solution $U(x)(w)$ of problem (16) is not available, and second, the infiniteness of the integral appearing in (23).

We propose the construction of the numerical solution by truncating the infinite integral and to perform to the numerical integration using composite Simpson's rule.

Given $X > 0$, $T > 0$, and $R > 0$, let us denote

$$I_1(X, T, R) = \int_0^R \{\varphi_1(X, w)P(w) + \varphi_2(X, w)Q(w)\} \sin(wT) dw, \quad (34)$$

$$J_1(X, w) = \int_0^X \varphi_1(s, w) ds, \quad J_2(X, w) = \int_0^X \varphi_2(s, w) ds, \quad (35)$$

$$I_2(X, T, R) = \int_0^R F(w) \sin(wT) \{J_1(X, w) + J_2(X, w)\} dw. \quad (36)$$

Given $R > 0$, take $r > 0$ and an integer N such that

$$r = \frac{R}{2N}, \quad w_j = jr, \quad j = 0, 1, \dots, 2N, \quad (37)$$

so that

$$w_{2N} = R. \quad (38)$$

Using composite Simpson's rule for computing integrals $I_i(X, T, R)$ for $i = 1, 2$, we have [7]

$$S_1(X, T, R, r) = \frac{r}{3} [\alpha_0 + 4\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4 + \dots + 2\alpha_{2N-2} + 4\alpha_{2N-1} + \alpha_{2N}], \quad (39)$$

$$\alpha_j = [\varphi_1(X, w_j)P(w_j) + \varphi_2(X, w_j)Q(w_j)] \sin(w_j T), \quad 0 \leq j \leq 2N, \quad (40)$$

$$S_2(X, T, R, r) = \frac{r}{3} [\beta_0 + 4\beta_1 + 2\beta_2 + 4\beta_3 + 2\beta_4 + \dots + 2\beta_{2N-2} + 4\beta_{2N-1} + \beta_{2N}], \quad (41)$$

$$\beta_j = [J_1(X, w_j) + J_2(X, w_j)] F(w_j) \sin(w_j T), \quad 0 \leq j \leq 2N. \quad (42)$$

Given $X > 0$, let $h > 0$ and let M be a positive integer so that

$$X = 2Mh, \quad h = \frac{X}{2M}, \quad s_l = lh, \quad 0 \leq l \leq 2M. \quad (43)$$

Now we approximate $J_i(X, w_j)$ for $i = 1, 2$, using composite Simpson's rule obtaining

$$J_1(X, w_j) = \frac{h}{3} [\gamma_0 + 4\gamma_1 + 2\gamma_2 + 4\gamma_3 + 2\gamma_4 + \dots + 2\gamma_{2M-2} + 4\gamma_{2M-1} + \gamma_{2M}], \quad (44)$$

$$J_2(X, w_j) = \frac{h}{3} [\rho_0 + 4\rho_1 + 2\rho_2 + 4\rho_3 + 2\rho_4 + \dots + 2\rho_{2M-2} + 4\rho_{2M-1} + \rho_{2M}], \quad (45)$$

$$\gamma_l = \varphi_1(s_l, w_j), \quad \rho_l = \varphi_2(s_l, w_j), \quad 0 \leq l \leq 2M. \quad (46)$$

Finally, in order to compute the values $\varphi_i(s_l, w_j)$, we use Stormer's method to the initial value problems [8, pp. 319–320]

$$\begin{aligned} \frac{d^2}{dx^2} \varphi_1(x, w_j) &= -c^2(x) w_j^2 \varphi_1(x, w_j), & 0 \leq x \leq X, \\ \varphi_1(0, w_j) &= 1, & \frac{d}{dx} \varphi_1(0, w_j) = 0, \end{aligned} \quad (47)$$

$$\begin{aligned} \frac{d^2}{dx^2} \varphi_2(x, w_j) &= -c^2(x) w_j^2 \varphi_2(x, w_j), & 0 \leq x \leq X, \\ \varphi_2(0, w_j) &= 0, & \frac{d}{dx} \varphi_2(0, w_j) = 1. \end{aligned} \quad (48)$$

Let $h > 0$ be given by (43) and let $\{\psi_{1j}(l), 0 \leq l \leq 2M\}$ be defined by

$$\begin{aligned} \psi_{1j}(l+2) - 2\psi_{1j}(l+1) + \psi_{1j}(l) &= h^2 c^2 ((l+1)h) w_j^2 \psi_{1j}(l+1), \\ \psi_{1j}(0) &= 1, \quad \psi_{1j}(1) = 1 \end{aligned} \quad (49)$$

and let $\{\psi_{2j}(l), 0 \leq l \leq 2M\}$ be defined by

$$\begin{aligned} \psi_{2j}(l+2) - 2\psi_{2j}(l+1) + \psi_{2j}(l) &= h^2 c^2 ((l+1)h) w_j^2 \psi_{2j}(l+1), \\ \psi_{2j}(0) &= 0, \quad \psi_{2j}(1) = h. \end{aligned} \quad (50)$$

Let $\tilde{\gamma}_l, \tilde{\rho}_l$ be defined by

$$\tilde{\gamma}_l = \psi_{1j}(l), \quad \tilde{\rho}_l = \psi_{2j}(l), \quad 0 \leq l \leq 2M, \quad (51)$$

and let $\tilde{J}_i(X, w_j), i = 1, 2$, be defined by

$$\tilde{J}_1(X, w_j) = \frac{h}{3} [\tilde{\gamma}_0 + 4\tilde{\gamma}_1 + 2\tilde{\gamma}_2 + 4\tilde{\gamma}_3 + 2\tilde{\gamma}_4 + \cdots + 2\tilde{\gamma}_{2M-2} + 4\tilde{\gamma}_{2M-1} + \tilde{\gamma}_{2M}], \quad (52)$$

$$\tilde{J}_2(X, w_j) = \frac{h}{3} [\tilde{\rho}_0 + 4\tilde{\rho}_1 + 2\tilde{\rho}_2 + 4\tilde{\rho}_3 + 2\tilde{\rho}_4 + \cdots + 2\tilde{\rho}_{2M-2} + 4\tilde{\rho}_{2M-1} + \tilde{\rho}_{2M}]. \quad (53)$$

Note that Stormer's method provides the approximation of $\varphi_1(X, w_j)$ given by $\psi_{1j}(2M)$ as well as the approximation of $\varphi_2(X, w_j)$ given by $\psi_{2j}(2M)$. Hence, let

$$\tilde{S}_1(X, T, R, r, h) = \frac{r}{3} [\tilde{\alpha}_0 + 4\tilde{\alpha}_1 + 2\tilde{\alpha}_2 + 4\tilde{\alpha}_3 + 2\tilde{\alpha}_4 + \cdots + 2\tilde{\alpha}_{2N-2} + 4\tilde{\alpha}_{2N-1} + \tilde{\alpha}_{2N}], \quad (54)$$

$$\tilde{\alpha}_j = [\psi_{1j}(2M)P(w_j) + \psi_{2j}(2M)Q(w_j)] \sin(w_j T), \quad 0 \leq j \leq 2N, \quad (55)$$

$$\tilde{S}_2(X, T, R, r, h) = \frac{r}{3} [\tilde{\beta}_0 + 4\tilde{\beta}_1 + 2\tilde{\beta}_2 + 4\tilde{\beta}_3 + 2\tilde{\beta}_4 + \cdots + 2\tilde{\beta}_{2N-2} + 4\tilde{\beta}_{2N-1} + \tilde{\beta}_{2N}], \quad (56)$$

$$\tilde{\beta}_j = [\tilde{J}_1(X, w_j) + \tilde{J}_2(X, w_j)] F(w_j) \sin(w_j T), \quad 0 \leq j \leq 2N. \quad (57)$$

Now, let us define the following numerical approximation:

$$\begin{aligned} u(X, T, R, r, h) &= \frac{2}{\pi} [\tilde{S}_1(X, T, R, r, h) + \tilde{S}_2(X, T, R, r, h)] \\ &= \frac{2r}{3\pi} \left[(\tilde{\alpha}_0 + \tilde{\beta}_0) + 4(\tilde{\alpha}_1 + \tilde{\beta}_1) + 2(\tilde{\alpha}_2 + \tilde{\beta}_2) + 4(\tilde{\alpha}_3 + \tilde{\beta}_3) + 2(\tilde{\alpha}_4 + \tilde{\beta}_4) + \cdots \right. \\ &\quad \left. + 4(\tilde{\alpha}_{2N-1} + \tilde{\beta}_{2N-1}) + (\tilde{\alpha}_{2N} + \tilde{\beta}_{2N}) \right], \end{aligned} \quad (58)$$

where $\tilde{\alpha}_j$ is given by (55) and $\tilde{\beta}_j$ by (57) with $\tilde{J}_i(X, w_j)$ defined by (52) for $i = 1$ and by (53) for $i = 2$.

By (52), (53), and (57), one gets

$$\begin{aligned} \tilde{\beta}_j &= F(w_j) \sin(w_j T) \frac{h}{3} [(\tilde{\gamma}_0 + \tilde{\rho}_0) + 4(\tilde{\gamma}_1 + \tilde{\rho}_1) + 2(\tilde{\gamma}_2 + \tilde{\rho}_2) + 4(\tilde{\gamma}_3 + \tilde{\rho}_3) \\ &\quad + 2(\tilde{\gamma}_4 + \tilde{\rho}_4) + \cdots + 4(\tilde{\alpha}_{2M-1} + \tilde{\rho}_{2M-1}) + (\tilde{\alpha}_{2M} + \tilde{\rho}_{2M})]. \end{aligned} \quad (59)$$

Summarizing, the following algorithm provides the numerical solution of problem (1)–(4).

1. INTEGRAL TRUNCATION.

- Given $X > 0$, $T > 0$, take $R > 0$ large.
- Take $r > 0$ and a positive integer N such that $r = R/2N$.
- Let $w_j = jr$ for $0 \leq j \leq 2N$.

2. SINE FOURIER TRANSFORMS EVALUATIONS.

- Evaluate $P(w_j)$, $Q(w_j)$, and $F(w_j)$ using (9),(37) for $0 \leq j \leq 2N$ with $h = p$, $h = q$, and $h = f$, respectively.
- Evaluate $\sin(w_j T)$ for $0 \leq j \leq 2N$.

3. NUMERICAL SOLUTION OF THE ASSOCIATED ORDINARY DIFFERENTIAL EQUATIONS.

- Let $h > 0$ and let M be a positive integer such that $h = X/2M$.
- Let $s_l = lh$ and evaluate $\psi_{1j}(l)$, $\psi_{2j}(l)$ using (49) and (50), respectively, for $0 \leq l \leq 2M$.

4. NUMERICAL SOLUTION OF THE PROBLEM AT (X, T) .

- Let $\tilde{\gamma}_l$, $\tilde{\rho}_l$ be given by (51) and evaluate $\tilde{\alpha}_j$ given by (55) and $\tilde{\beta}_j$ by (59) for $0 \leq l \leq 2M$.
- Compute $\tilde{S}_1(X, T, R, r, h)$ and $\tilde{S}_2(X, T, R, r, h)$ given by (54) and (56), respectively.
- Compute $u(X, T, R, r, h)$ given by (58).

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